Disentangling positivity constraints for generalized parton distributions

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Positivity constraints are derived for the generalized parton distributions (GPDs) of spin-1/2 hadrons. The analysis covers the full set of eight twist-2 GPDs. Several new inequalities are obtained which constrain GPDs by various combinations of usual (forward) unpolarized and polarized parton distributions, including the transversity distribution.

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I. INTRODUCTION

It was shown during the last decade that a number of hard exclusive processes, including deeply virtual Compton scattering and hard exclusive meson production, are calculable in QCD in terms of generalized parton distributions (GPDs) also known as skewed, off-forward, or nondiagonal [1–5]. GPDs cannot be measured directly: the cross sections of hard exclusive processes give us information only about some integrals containing combinations of GPDs with other perturbative and nonperturbative functions. Therefore any model independent theoretical constraints on GPDs are of interest.

The aim of this paper is to obtain new theoretical restrictions on GPDs in terms of the usual (forward) parton distributions using the so-called positivity bounds. The positivity bounds for GPDs are derived by taking the square of the superposition of parton-hadron states [6–9]:

$$\left\| \sum_{k=1}^{2} \sum_{\lambda\mu} c_{\lambda\mu}^{(k)} \int \frac{d\tau}{2\pi} e^{i\tau x_k (P_k n)} \phi_\mu(\tau n) |P_k,\lambda\rangle \right\|^2 \geqslant 0. \quad (1)$$

Here $|P_k,\lambda\rangle$ is the nucleon state with momentum P_k and polarization λ, ϕ_μ is the component of the quark (or gluon) field corresponding to the polarization μ, and n is a light-cone vector.

Expanding the left-hand side (LHS) of inequality (1) one obtains an expression containing diagonal terms corresponding to the usual parton distributions and nondiagonal terms corresponding to GPDs. Generally speaking, inequality (1) should hold for any coefficients $c_{\lambda\mu}^{(k)}$ but so far it has been analyzed only for some specific sets of $c_{\lambda\mu}^{(k)}$. As a result the positivity bounds for GPDs derived earlier are not as strong as they could be.

In particular, an interesting inequality for GPD H^q was derived in Refs. [7,8]

$$H^q(x,\xi,t) \leqslant \sqrt{\frac{q(x_1)q(x_2)}{1-\xi^2}}. \quad (2)$$

Here the notation of X. Ji [9] is used for the quark GPDs ($H^q, E^q, \ldots$) and for their arguments x,ξ,t. In the right-hand side (RHS) of Eq. (2), $q(x_k)$ is the usual (forward) quark unpolarized distribution taken at values

$$x_1 = \frac{x+\xi}{1+\xi}, \quad x_2 = \frac{x-\xi}{1-\xi}. \quad (3)$$

In Ref. [10] it was noticed that in the original derivation of inequality (2) the contribution of GPD E^q was ignored. The authors of Ref. [10] have derived the correct version of inequality (2)

$$\left| H^q(x,\xi,t) - \frac{\xi^2}{1-\xi^2} E^q(x,\xi,t) \right| \leqslant \sqrt{\frac{q(x_1)q(x_2)}{1-\xi^2}}. \quad (4)$$

Later in Ref. [11] a stronger bound for H^q and E^q was obtained:

$$\left[H^q(x,\xi,t) - \frac{\xi^2}{1-\xi^2} E^q(x,\xi,t) \right]^2 + \left[\frac{\sqrt{t_0-t}}{2m\sqrt{1-\xi^2}} E^q(x,\xi,t) \right]^2 \leqslant \frac{q(x_1)q(x_2)}{1-\xi^2}. \quad (5)$$

Here m is the nucleon mass and t_0 is the maximal (negative) value of the squared nucleon momentum transfer t:

$$t_0 = -\frac{4\xi^2 m^2}{1-\xi^2}, \quad t \leqslant t_0 \leqslant 0. \quad (6)$$

One has to keep in mind that the general positivity bound (1) contains much more constraints on GPDs than the above examples (4), (5). The aim of this paper is to extract from inequality (1) as much information as possible.

In Ref. [12] a classification of twist-2 GPDs was suggested in terms of eight distributions

$$H, E, H_T, E_T, \tilde{H}, \tilde{E}, \tilde{H}_T, \tilde{E}_T. \quad (7)$$

Inequality (1) defines some domain in the eight-dimensional space of GPDs (7). In this paper this "allowed region" is studied.

The paper is organized as follows. In Sec. II the positivity condition is formulated in terms of helicity amplitudes. In Sec. III we find the region in the eight-dimensional space of twist-two GPDs which is allowed by the positivity constraints. In Sec. IV we show how to project this eight-dimensional "allowed region" onto one axis corresponding to a single GPD, then we derive the general inequality which can be applied to any GPD and to any linear combination of GPDs. In Sec. V this general inequality is written in an explicit form for various quark GPDs. The main part of the paper is devoted to the quark GPDs. In Sec. VI similar inequalities for gluon GPDs are presented.

II. POSITIVITY BOUND IN TERMS OF HELICITY AMPLITUDES

It is convenient to analyze the general positivity inequality (1) using ''helicity amplitudes'' (the term is not quite justified but it is used for brevity) introduced in Ref. [12]

$$A_{\lambda'\mu',\lambda\mu}=\int\frac{dz^-}{2\pi} \times e^{ixP^+z^-}\langle p',\lambda'|O_{\mu',\mu}(z)|p,\lambda\rangle|_{z^+=0,z^\perp=0}. \tag{8}$$

Here $O_{\mu',\mu}$ are bilinear quark light-ray operators with the polarization indices μ,μ'. Next, $|p,\lambda\rangle$ is a nucleon state with momentum p and polarization λ (in the sense of light-cone helicity states [13]), P^+ is the light-cone component of vector $P=(p+p')/2$. The explicit expressions for $A_{\lambda'\mu',\lambda\mu}$ in terms of GPDs (7) are listed in the Appendix.

With two values for each polarization index of $A_{\lambda'\mu',\lambda\mu}$ we have $2^4=16$ components but due to the parity invariance [12]

$$A_{-\lambda',-\mu';-\lambda,-\mu}=(-1)^{\lambda'-\mu'-\lambda+\mu}A_{\lambda'\mu',\lambda\mu} \tag{9}$$

only eight components are independent. This is exactly the number of twist-2 GPDs (7). There is a one-to-one linear correspondence between the independent components of helicity amplitudes $A_{\lambda'\mu',\lambda\mu}$ and GPDs (7). Therefore we can study the positivity bound (1) using helicity amplitudes $A_{\lambda'\mu',\lambda\mu}$. The result will be expressed in terms of usual GPDs $H,E,\ldots$ (7) at the final stage of the work. The helicity amplitudes $A_{\lambda'\mu',\lambda\mu}$ are useful for the derivation of the positivity bounds because the underlying inequality (1) looks quite simple in terms of $A_{\lambda'\mu',\lambda\mu}$:

$$c_M^{(1)*}A_{MN}^\dagger(x,\xi,t)c_N^{(2)}+c_M^{(2)*}A_{MN}(x,\xi,t)c_N^{(1)} +c_M^{(1)*}A_{MN}(x_1,0,0)c_N^{(1)} +c_M^{(2)*}A_{MN}(x_2,0,0)c_N^{(2)}\geqslant 0. \tag{10}$$

Here we combined the nucleon-quark polarization indices λ,μ into one multiindex $M=(\lambda,\mu)$. The dagger in $A_{MN}^\dagger$ stands for the Hermitean conjugation. As usual the summation over repeated indices is implied. Parameters x_1,x_2 appearing in $A_{MN}(x_{1,2},0,0)$ are connected to arguments x,ξ of $A_{MN}(x,\xi,t)$ by relations (3) and the region $x>|\xi|$ is implied.

In this paper inequality (10) is analyzed in its general form with arbitrary coefficients $c_M^{(k)}$. Allowing arbitrary coefficients $c_M^{(k)}$ we extend the analysis to all kinds of GPDs of leading twist including those with the quark helicity flip. It is interesting that by including the GPDs with the quark helicity flip in the analysis we get new constraints on helicity non-flip GPDs.

Inequality (10) that is valid for arbitrary coefficients $c_M^{(k)}$ is nothing else but the positivity condition for the following 8×8 matrix:

$$\begin{pmatrix} A(x_1,0,0) & A^\dagger(x,\xi,t)\\ A(x,\xi,t) & A(x_2,0,0)\end{pmatrix}\geqslant 0. \tag{11}$$

In the next section we shall transform this positivity condition to a form more convenient for the practical work with GPDs.

III. SOLUTION OF POSITIVITY CONSTRAINT

The first step is to reduce the analysis of the positivity of the 8×8 matrix (11) to a problem involving only 4×4 matrices. This can be done using the symmetry property (9). This symmetry allows us to make matrix A_{MN} block diagonal. Performing the transformation

$$c_{++}^{(k)}=\tilde{c}_1^{(k)}+\tilde{c}_3^{(k)},\quad c_{--}^{(k)}=\tilde{c}_1^{(k)}-\tilde{c}_3^{(k)}, \tag{12}$$

$$c_{+-}^{(k)}=\tilde{c}_2^{(k)}+\tilde{c}_4^{(k)},\quad c_{-+}^{(k)}=\tilde{c}_4^{(k)}-\tilde{c}_2^{(k)}, \tag{13}$$

we obtain

$$c_M^{(k)*}A_{MN}c_N^{(l)}=\tilde{c}_a^{(k)*}\tilde{A}_{ab}\tilde{c}_b^{(l)}. \tag{14}$$

where the 4×4 matrix $\tilde{A}_{ab}$ consists of two 2×2 blocks B_1,B_2

$$\tilde{A}=\begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}. \tag{15}$$

The explicit expressions for matrix elements A_{MN} and $\tilde{A}_{ab}$ are given in the Appendix. Since the 4×4 matrix $\tilde{A}$ consists of two 2×2 blocks it has eight nonzero components and this is exactly the number of all twist-2 GPDs (7).

In this representation the positivity condition (11) results in two independent positivity constraints for 4×4 matrices

$$C_s\equiv\begin{pmatrix} B_s(x_1,0,0) & B_s^\dagger(x,\xi,t)\\ B_s(x,\xi,t) & B_s(x_2,0,0)\end{pmatrix}\geqslant 0\quad (s=1,2). \tag{16}$$

The positivity of matrix C_s automatically leads to the positivity of its diagonal blocks

$$F_s(x_k)=B_s(x_k,0,0)\geqslant 0. \tag{17}$$

Matrices F_s are diagonal and correspond to the following combinations of usual forward distributions:

$$F_s=\begin{pmatrix} q+\Delta_L q+2\varepsilon_s\Delta_T q & 0\\ 0 & q-\Delta_L q\end{pmatrix} \tag{18}$$

$$(\varepsilon_1=-\varepsilon_2=1).$$

Here $q,\Delta_L q,\Delta_T q$ are, respectively, unpolarized, longitudinally polarized, and transversity quark distributions. Thus we reproduce the standard positivity bounds on parton distributions

$$|\Delta_L q| \leqslant q, \tag{19}$$

including the Soffer inequality [14]

$$2|\Delta_T q| \leqslant q + \Delta_L q. \tag{20}$$

Now we can replace Eq. (16) by the following equivalent condition:

$$\begin{pmatrix} F_s^{-1/2}(x_1) & 0 \\ 0 & F_s^{-1/2}(x_2) \end{pmatrix} C_s \begin{pmatrix} F_s^{-1/2}(x_1) & 0 \\ 0 & F_s^{-1/2}(x_2) \end{pmatrix} = \begin{pmatrix} 1 & K_s^+ \\ K_s & 1 \end{pmatrix} \geqslant 0, \tag{21}$$

where

$$K_s = F_s^{-1/2}(x_2) B_s(x,\xi,t) F_s^{-1/2}(x_1). \tag{22}$$

It is straightforward to check that positivity condition (21) is equivalent to the following bound on the norm of matrix K_s:

$$\|K_s\| \leqslant 1. \tag{23}$$

Equivalently this can be reformulated as the positivity of the matrix

$$1 - K_s^\dagger K_s \geqslant 0 \tag{24}$$

or in terms of matrix elements

$$\sum_j |(K_s)_{ij}|^2 \leqslant 1, \quad \sum_i |(K_s)_{ij}|^2 \leqslant 1. \tag{25}$$

Using the trace and the determinant of matrix $K_s^\dagger K_s$ we can rewrite these constraints as follows:

$$\mathrm{Tr}\, K_s^\dagger K_s \leqslant 2, \quad \mathrm{Tr}\, K_s^\dagger K_s \leqslant 1 + \det K_s^\dagger K_s. \tag{26}$$

In our case of matrix K_s (22) these conditions take the form

$$\mathrm{Tr}[B_s^\dagger(x,\xi,t) F_s^{-1}(x_2) B_s(x,\xi,t) F_s^{-1}(x_1)] \leqslant 2, \tag{27}$$

$$\mathrm{Tr}[B_s^\dagger(x,\xi,t) F_s^{-1}(x_2) B_s(x,\xi,t) F_s^{-1}(x_1)] \leqslant 1 + \frac{|\det B_s(x,\xi,t)|^2}{\det F_s(x_1) \det F_s(x_2)}. \tag{28}$$

We note that the matrix $B_s(x,\xi,t)$ is composed of the linear combinations of various GPDs. The explicit expressions for the matrix elements of B_s in terms of GPDs (7) can be read from (15) using the matrix elements $\tilde{A}_{ab}$ listed in the Appendix. Inserting these expressions into inequalities (27), (28) one can write explicit bounds on GPDs (7). Note that the trace $\mathrm{Tr}[B_s^\dagger F_s^{-1}(x_2) B_s F_s^{-1}(x_1)]$ is quadratic in GPDs and the squared determinant $|\det B_s|^2$ is quartic. Thus inequalities (27), (28) are polynomial inequalities for GPDs of order 2 and 4, respectively. However, instead of dealing with explicit expressions containing polynomials of eight GPDs it is much more convenient to work with the compact form of the positivity condition (23).

IV. BOUNDS ON SEPARATE GPDS AND THEIR LINEAR COMBINATIONS

With eight independent GPDs hidden in matrices $B_s(x,\xi,t)$ the above constraints (27), (28) are hardly tractable. Therefore it is interesting to derive bounds for single GPDs. Since matrix elements of $B_s(x,\xi,t)$ are linear in GPDs we can represent any GPD $G = H, E, \ldots$ (7) in the form

$$G(x,\xi,t) = \sum_{s=1}^{2} \mathrm{Tr}[L_s^{(G)} B_s(x,\xi,t)], \tag{29}$$

where $L_s^{(G)}$ are 2×2 matrices depending on the specific choice of the GPD G. Hence the derivation of the bounds for single GPDs reduces to the problem of the calculation of the maximum

$$|G(x,\xi,t)| \leqslant \sum_{s=1}^{2} \max_{B_s} |\mathrm{Tr}[L_s^{(G)} B_s(x,\xi,t)]| \tag{30}$$

under constraints (27), (28). Using relation (22) we can rewrite this as follows:

$$\max_{B_s} |\mathrm{Tr}[L_s B_s(x,\xi,t)]| = \max_{K_s : \|K_s\| \leqslant 1} |\mathrm{Tr}[F_s^{1/2}(x_1) L_s F_s^{1/2}(x_2) K_s]|, \tag{31}$$

where the maximum in the RHS is taken with respect to matrices K_s obeying the constraint $\|K_s\| \leqslant 1$. The general solution of this problem for an arbitrary matrix M is

$$\max_{K:\|K\| \leqslant 1} |\mathrm{Tr}(KM)| = \mathrm{Tr}[(M^\dagger M)^{1/2}]. \tag{32}$$

Here $(M^\dagger M)^{1/2}$ should be understood as a function of the matrix. Applying the general result (32) to our case (31) we find

$$|G(x,\xi,t)| \leqslant \sum_{s=1}^{2} \mathrm{Tr}\{[F_s(x_1) L_s^{(G)} F_s(x_2) L_s^{(G)\dagger}]^{1/2}\}. \tag{33}$$

This inequality can be used for any GPD and any linear combination of GPDs.

V. BOUNDS ON QUARK GPDS

A. Inequalities for $H^q - \xi^2(1-\xi^2)^{-1} E^q$

As an example let us consider a linear combination of GPDs H^q and E^q which has a simple expression in terms of amplitudes $\tilde{A}_{ab}$ (15). Using the equations of the Appendix we find

$$H^q - \frac{\xi^2}{1-\xi^2} E^q = \frac{1}{4\sqrt{1-\xi^2}} (\tilde{A}_{11} + \tilde{A}_{22} + \tilde{A}_{33} + \tilde{A}_{44}). \tag{34}$$

This structure corresponds to the following choice of matrices $L_s^{(G)}$ in Eq. (29):

$$L_1^{(G)}=L_2^{(G)}=\frac{1}{4\sqrt{1-\xi^2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \qquad (35)$$

With these matrices $L_s^{(G)}$ we obtain from inequality (33):

$$\left|H^q(x,\xi,t)-\frac{\xi^2}{1-\xi^2}E^q(x,\xi,t)\right| \leqslant \frac{1}{4\sqrt{1-\xi^2}}\sum_{s=1}^{2}\mathrm{Tr}\{[F_s(x_1)F_s(x_2)]^{1/2}\}. \qquad (36)$$

Below for brevity we omit the arguments x,ξ,t of GPDs and write the arguments x_1,x_2 of the forward distributions as subscripts.

Using expression (18) for $F_s(x_k)$ we can rewrite inequality (36) in the form

$$\left|H^q-\frac{\xi^2}{1-\xi^2}E^q\right| \leqslant \frac{1}{4\sqrt{1-\xi^2}}\{\sqrt{(q+\Delta_Lq-2\Delta_Tq)_{x_1}(q+\Delta_Lq-2\Delta_Tq)_{x_2}} + \sqrt{(q+\Delta_Lq+2\Delta_Tq)_{x_1}(q+\Delta_Lq+2\Delta_Tq)_{x_2}} + 2\sqrt{(q-\Delta_Lq)_{x_1}(q-\Delta_Lq)_{x_2}}\}. \qquad (37)$$

One can obtain a weaker bound by maximizing the RHS with respect to Δ_Tq in the range allowed by the Soffer inequality (20)

$$\left|H^q-\frac{\xi^2}{1-\xi^2}E^q\right| \leqslant \frac{1}{2\sqrt{1-\xi^2}}\{\sqrt{(q+\Delta_Lq)_{x_1}(q+\Delta_Lq)_{x_2}} + \sqrt{(q-\Delta_Lq)_{x_1}(q-\Delta_Lq)_{x_2}}\}. \qquad (38)$$

Another still weaker inequality can be derived by maximizing the RHS of Eq. (38) with respect to Δ_Lq

$$\left|H^q-\frac{\xi^2}{1-\xi^2}E^q\right| \leqslant \sqrt{\frac{q(x_1)q(x_2)}{1-\xi^2}}. \qquad (39)$$

We have reproduced inequality (4).

Using this strategy one can obtain bounds for any GPD or any linear combination of GPDs starting from the general inequality (33). From the above example we see that there is a hierarchy of inequalities: (1) strong inequalities where GPDs are bounded by combinations of all forward quark distributions including the transversity distribution; (2) weaker inequalities without the forward transversity distribution; (3) still weaker inequalities where GPDs are bounded by only the unpolarized forward distribution.

B. Inequalities for $\tilde{H}^q-\xi^2(1-\xi^2)^{-1}\tilde{E}^q$

Now let us turn to the combination of GPDs

$$\tilde{H}^q-\frac{\xi^2}{1-\xi^2}\tilde{E}^q=\frac{1}{4\sqrt{1-\xi^2}}(\tilde{A}_{11}-\tilde{A}_{22}+\tilde{A}_{33}-\tilde{A}_{44}). \qquad (40)$$

This corresponds to the following matrices $L_s^{(G)}$ in Eq. (29):

$$L_1^{(G)}=L_2^{(G)}=\frac{1}{4\sqrt{1-\xi^2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad (41)$$

Inserting these matrices $L_1^{(G)},L_2^{(G)}$ into the general inequality (33) we arrive at the same expression in the RHS as in the above section where the bounds for $H^q-\xi^2(1-\xi^2)^{-1}E^q$ were derived. Therefore we have the same inequalities for $H^q-\xi^2(1-\xi^2)^{-1}E^q$ and $\tilde{H}^q-\xi^2(1-\xi^2)^{-1}\tilde{E}^q$:

$$\left|\tilde{H}^q-\frac{\xi^2}{1-\xi^2}\tilde{E}^q\right| \leqslant \frac{1}{4\sqrt{1-\xi^2}}\{\sqrt{(q+\Delta_Lq-2\Delta_Tq)_{x_1}(q+\Delta_Lq-2\Delta_Tq)_{x_2}} + \sqrt{(q+\Delta_Lq+2\Delta_Tq)_{x_1}(q+\Delta_Lq+2\Delta_Tq)_{x_2}} + 2\sqrt{(q-\Delta_Lq)_{x_1}(q-\Delta_Lq)_{x_2}}\}, \qquad (42)$$

$$\left|\tilde{H}^q-\frac{\xi^2}{1-\xi^2}\tilde{E}^q\right| \leqslant \frac{1}{2\sqrt{1-\xi^2}}\{\sqrt{(q+\Delta_Lq)_{x_1}(q+\Delta_Lq)_{x_2}} + \sqrt{(q-\Delta_Lq)_{x_1}(q-\Delta_Lq)_{x_2}}\}, \qquad (43)$$

$$\left|\tilde{H}^q-\frac{\xi^2}{1-\xi^2}\tilde{E}^q\right| \leqslant \sqrt{\frac{q(x_1)q(x_2)}{1-\xi^2}}. \qquad (44)$$

C. Inequalities for E^q

In the derivation of inequalities for E^q one deals with nondiagonal matrices $L_1^{(G)},L_2^{(G)}$ in the RHS of Eq. (33). Computing the trace in the RHS of Eq. (33) one obtains the following bound:

$$|E^q| \leqslant \frac{m}{2\sqrt{t_0-t}}\{\sqrt{(q-\Delta_Lq)_{x_1}(q+\Delta_Lq+2\Delta_Tq)_{x_2}} + \sqrt{(q+\Delta_Lq+2\Delta_Tq)_{x_1}(q-\Delta_Lq)_{x_2}} + \sqrt{(q+\Delta_Lq-2\Delta_Tq)_{x_1}(q-\Delta_Lq)_{x_2}} + \sqrt{(q-\Delta_Lq)_{x_1}(q+\Delta_Lq-2\Delta_Tq)_{x_2}}\}. \qquad (45)$$

Maximizing the RHS with respect to Δ_Tq we arrive at a weaker bound

$$|E^q| \leqslant \frac{m}{\sqrt{t_0-t}}\left[\sqrt{(q+\Delta_L q)_{x_1}(q-\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q+\Delta_L q)_{x_2}}\right]. \tag{46}$$

Next one can maximize the RHS of Eq. (46) with respect $\Delta_L q$

$$|E^q| \leqslant \frac{2m}{\sqrt{t_0-t}}\sqrt{q(x_1)q(x_2)}. \tag{47}$$

This coincides with the bound for E^q derived earlier in Ref. [10] up to a typo there [15]. This bound also can be obtained directly from inequality (5).

D. Inequalities for $\tilde{E}^q$

Inequalities for $\tilde{E}^q$ differ from inequalities for E^q only by a factor of ξ^{-1} in the RHS:

$$|\tilde{E}^q| \leqslant \frac{m}{2\xi\sqrt{t_0-t}}\left\{\sqrt{(q-\Delta_L q)_{x_1}(q+\Delta_L q+2\Delta_T q)_{x_2}} + \sqrt{(q+\Delta_L q+2\Delta_T q)_{x_1}(q-\Delta_L q)_{x_2}} + \sqrt{(q+\Delta_L q-2\Delta_T q)_{x_1}(q-\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q+\Delta_L q-2\Delta_T q)_{x_2}}\right\}, \tag{48}$$

$$|\tilde{E}^q| \leqslant \frac{m}{\xi\sqrt{t_0-t}}\left[\sqrt{(q+\Delta_L q)_{x_1}(q-\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q+\Delta_L q)_{x_2}}\right], \tag{49}$$

$$|\tilde{E}^q| \leqslant \frac{2m}{\xi\sqrt{t_0-t}}\sqrt{q(x_1)q(x_2)}. \tag{50}$$

E. Inequalities for H^q

Using Eq. (33) one can derive the following bound for H^q:

$$|H^q| \leqslant \tfrac{1}{4}\{\alpha^2[\sqrt{f_1(x_1)f_1(x_2)} + \sqrt{f_2(x_1)f_2(x_2)}]^2 + \beta^2[\sqrt{f_1(x_1)f_2(x_2)} + \sqrt{f_2(x_1)f_1(x_2)}]^2\}^{1/2} + \tfrac{1}{4}\{\alpha^2[\sqrt{f_3(x_1)f_3(x_2)} + \sqrt{f_2(x_1)f_2(x_2)}]^2 + \beta^2[\sqrt{f_3(x_1)f_2(x_2)} + \sqrt{f_2(x_1)f_3(x_2)}]^2\}^{1/2}. \tag{51}$$

Here we use a compact notation for the combinations of the usual (forward) parton distributions

$$f_1 = q+\Delta_L q+2\Delta_T q, \quad f_2 = q-\Delta_L q, \quad f_3 = q+\Delta_L q-2\Delta_T q \tag{52}$$

which appear in the diagonal matrices F_s (18). Parameters α,β are defined as follows:

$$\alpha = \frac{1}{\sqrt{1-\xi^2}}, \quad \beta = \frac{\xi^2}{1-\xi^2}\frac{2m}{\sqrt{t_0-t}}. \tag{53}$$

The weaker bound without transversity distribution $\Delta_T q$ looks as follows:

$$|H^q| \leqslant \tfrac{1}{2}\{\alpha^2[\sqrt{(q+\Delta_L q)_{x_1}(q+\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q-\Delta_L q)_{x_2}}]^2 + \beta^2[\sqrt{(q+\Delta_L q)_{x_1}(q-\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q+\Delta_L q)_{x_2}}]^2\}^{1/2}. \tag{54}$$

Maximizing the RHS with respect to $\Delta_L q$ we obtain

$$|H^q| \leqslant \sqrt{\left(1+\frac{-t_0\xi^2}{t_0-t}\right)\frac{q(x_1)q(x_2)}{1-\xi^2}}. \tag{55}$$

Note that inequality (55) is stronger than the bound for H^q derived in [11] but still weaker than the obsolete inequality (2).

F. Inequalities for $\tilde{H}^q$

Similarly we find for $\tilde{H}^q$

$$|\tilde{H}^q| \leqslant \frac{1}{4}\left\{\alpha^2[\sqrt{f_1(x_1)f_1(x_2)} + \sqrt{f_2(x_1)f_2(x_2)}]^2 + \left(\frac{\beta}{\xi}\right)^2[\sqrt{f_1(x_1)f_2(x_2)} + \sqrt{f_2(x_1)f_1(x_2)}]^2\right\}^{1/2} + \frac{1}{4}\left\{\alpha^2[\sqrt{f_3(x_1)f_3(x_2)} + \sqrt{f_2(x_1)f_2(x_2)}]^2 + \left(\frac{\beta}{\xi}\right)^2[\sqrt{f_3(x_1)f_2(x_2)} + \sqrt{f_2(x_1)f_3(x_2)}]^2\right\}^{1/2}. \tag{56}$$

Parameters α,β are given in Eqs. (53) and functions f_k are the same as in Eqs. (52). The weaker inequality without the transversity distribution is

$$|\tilde{H}^q| \leqslant \frac{1}{2}\left\{\alpha^2[\sqrt{(q+\Delta_L q)_{x_1}(q+\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q-\Delta_L q)_{x_2}}]^2 + \left(\frac{\beta}{\xi}\right)^2[\sqrt{(q+\Delta_L q)_{x_1}(q-\Delta_L q)_{x_2}} + \sqrt{(q-\Delta_L q)_{x_1}(q+\Delta_L q)_{x_2}}]^2\right\}^{1/2} \tag{57}$$

and the bound without forward polarized distributions looks as follows:

$$|\tilde{H}^q| \leq \sqrt{\frac{-t}{t_0 - t} \frac{q(x_1) q(x_2)}{1-\xi^2}}. \tag{58}$$

VI. INEQUALITIES FOR GLUON GPDS

One can easily generalize the above inequalities for the case of gluon GPDs. As noted in Ref. [12] helicity amplitudes $A_{\lambda'\mu',\lambda\mu}$ without parton helicity flip (i.e., with $\mu = \mu'$) are represented by the same expressions in terms of GPDs in the quark and gluon cases. As a result the expressions for $H, E, \tilde{H}, \tilde{E}$ in terms of amplitudes $\tilde{A}_{ab}$ also have the same form for quarks and gluons.

The difference between the quark and gluon cases comes from the following sources.

(1) In the gluon case the quark inequality (10) should be replaced by

$$\frac{1}{\sqrt{1-\xi^2}} [c_M^{(1)*} A^\dagger_{MN}(x,\xi,t) c_N^{(2)} + c_M^{(2)*} A_{MN}(x,\xi,t) c_N^{(1)}] + c_M^{(1)*} A_{MN}(x_1,0,0) c_N^{(1)} + c_M^{(2)*} A_{MN}(x_2,0,0) c_N^{(2)} \geq 0. \tag{59}$$

(2) The standard definition of the *forward* gluon distribution contains an extra factor of x compared to the quark distributions.

(3) We use the normalization conventions of Ref. [12] for gluon GPDs so that the forward limit of gluon GPDs differs from the standard forward gluon unpolarized g and polarized $\Delta_L g$ distributions by a factor of x:

$$H^g(x,0,0) = x g(x), \quad \tilde{H}^g(x,0,0) = x \Delta_L g(x). \tag{60}$$

Combining all above factors we see that the transition from the quark case to the gluon one reduces to the replacement

$$\sqrt{q_1(x_1) q_2(x_2)} \to \sqrt{1-\xi^2} \sqrt{x_1 x_2} \sqrt{g_1(x_1) g_2(x_2)} = \sqrt{x^2-\xi^2} \sqrt{g_1(x_1) g_2(x_2)}, \tag{61}$$

where $g_k = g, \Delta_L g$ are gluon analogues of quark distributions $q_k = q, \Delta_L q$. Of course, one has to keep in mind that the forward gluon transversity distribution vanishes in the case of spin-1/2 hadrons which is considered here.

As a result we obtain

$$\left| H^g - \frac{\xi^2}{1-\xi^2} E^g \right| \leq \frac{1}{2} \sqrt{\frac{x^2-\xi^2}{1-\xi^2}} [\sqrt{(g+\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g-\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}], \tag{62}$$

$$\left| H^g - \frac{\xi^2}{1-\xi^2} E^g \right| \leq \sqrt{\frac{x^2-\xi^2}{1-\xi^2} g(x_1) g(x_2)}, \tag{63}$$

$$\left| \tilde{H}^g - \frac{\xi^2}{1-\xi^2} \tilde{E}^g \right| \leq \frac{1}{2} \sqrt{\frac{x^2-\xi^2}{1-\xi^2}} [\sqrt{(g+\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g-\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}], \tag{64}$$

$$\left| \tilde{H}^g - \frac{\xi^2}{1-\xi^2} \tilde{E}^g \right| \leq \sqrt{\frac{x^2-\xi^2}{1-\xi^2} g(x_1) g(x_2)}, \tag{65}$$

$$|E^g| \leq \frac{m}{\sqrt{t_0-t}} \sqrt{x^2-\xi^2} [\sqrt{(g-\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g+\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}], \tag{66}$$

$$|E^g| \leq \frac{2m\sqrt{x^2-\xi^2}}{\sqrt{t_0-t}} \sqrt{g(x_1) g(x_2)}, \tag{67}$$

$$|\tilde{E}^g| \leq \frac{m\sqrt{x^2-\xi^2}}{\xi\sqrt{t_0-t}} [\sqrt{(g-\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g+\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}], \tag{68}$$

$$|\tilde{E}^g| \leq \frac{2m\sqrt{x^2-\xi^2}}{\xi\sqrt{t_0-t}} \sqrt{g(x_1) g(x_2)}, \tag{69}$$

$$|H^g| \leq \frac{\sqrt{x^2-\xi^2}}{2} \{ \alpha^2 [\sqrt{(g+\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g-\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}]^2 + \beta^2 [\sqrt{(g-\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g+\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}]^2 \}^{1/2}, \tag{70}$$

$$|H^g| \leq \sqrt{x^2-\xi^2} \sqrt{\left(1 + \frac{-t_0 \xi^2}{t_0-t}\right) \frac{g(x_1) g(x_2)}{1-\xi^2}}, \tag{71}$$

$$|\tilde{H}^g| \leq \frac{\sqrt{x^2-\xi^2}}{2} \left\{ \alpha^2 [\sqrt{(g+\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g-\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}]^2 + \left(\frac{\beta}{\xi}\right)^2 [\sqrt{(g-\Delta_L g)_{x_1} (g+\Delta_L g)_{x_2}} + \sqrt{(g+\Delta_L g)_{x_1} (g-\Delta_L g)_{x_2}}]^2 \right\}^{1/2}, \tag{72}$$

$$|\tilde{H}^g| \leq \sqrt{x^2-\xi^2} \sqrt{\frac{-t}{t_0-t} \frac{g(x_1) g(x_2)}{1-\xi^2}}. \tag{73}$$

Parameters α, β are given by Eqs. (53). Inequality (63) was derived earlier in Ref. [10].

VII. CONCLUSIONS

In this paper we found the domain in the eight-dimensional space of twist-2 GPDs (7) which is allowed by the positivity constraint (1). This region is described by polynomial inequalities (27), (28). These inequalities mix various GPDs in a nontrivial way. If one is interested in bounds on a ingle GPD then one has to project this eight-dimensional domain onto the axis corresponding to the chosen GPD. This projection leads to the general inequality (33) which can be applied to any GPD or any linear combination of GPDs. The explicit inequalities for various quark GPDs are presented in Sec. V and for the gluon GPDs in Sec. VI.

The derivation of the positivity bounds presented here ignored the problem of the renormalization. It is well known that in the case of forward parton distributions the positivity can be violated at low normalization points where parton distributions can essentially differ from the corresponding structure functions associated with physical cross sections. Similarly the positivity bounds derived for GPDs in this paper can be also violated at low normalization points. In the case of forward parton distributions the one-loop evolution to higher normalization points is known to preserve all positivity properties. For some special inequalities for GPDs the stability with respect to the evolution was analyzed in Ref. [8]. In Ref. [16] the one-loop evolution stability was demonstrated for a wider class of positivity bounds including the inequalities considered in this paper.

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APPENDIX: HELICITY AMPLITUDES

Below we list the expressions for the amplitudes $A_{\lambda'\mu',\lambda\mu}$ (8) in terms of GPDs (7). The derivation and the details can be found in Ref. [12]:

$$A^q_{++,++}=\sqrt{1-\xi^2}\left(\frac{H^q+\tilde{H}^q}{2}-\frac{\xi^2}{1-\xi^2}\,\frac{E^q+\tilde{E}^q}{2}\right), \quad \text{(A1)}$$

$$A^q_{-+,-+}=\sqrt{1-\xi^2}\left(\frac{H^q-\tilde{H}^q}{2}-\frac{\xi^2}{1-\xi^2}\,\frac{E^q-\tilde{E}^q}{2}\right), \quad \text{(A2)}$$

$$A^q_{++,-+}=-\varepsilon\frac{\sqrt{t_0-t}}{2m}\,\frac{E^q-\xi\tilde{E}^q}{2}, \quad \text{(A3)}$$

$$A^q_{-+,++}=\varepsilon\frac{\sqrt{t_0-t}}{2m}\,\frac{E^q+\xi\tilde{E}^q}{2}, \quad \text{(A4)}$$

$$A^q_{++,+-}=\varepsilon\frac{\sqrt{t_0-t}}{2m}\left(\tilde{H}^q_T+(1-\xi)\,\frac{E^q_T+\tilde{E}^q_T}{2}\right), \quad \text{(A5)}$$

$$A^q_{-+,--}=\varepsilon\frac{\sqrt{t_0-t}}{2m}\left(\tilde{H}^q_T+(1+\xi)\,\frac{E^q_T-\tilde{E}^q_T}{2}\right), \quad \text{(A6)}$$

$$A^q_{++,--}=\sqrt{1-\xi^2}\left(H^q_T+\frac{t_0-t}{4m^2}\tilde{H}^q_T-\frac{\xi^2}{1-\xi^2}E^q_T+\frac{\xi}{1-\xi^2}\tilde{E}^q_T\right), \quad \text{(A7)}$$

$$A^q_{-+,+-}=-\sqrt{1-\xi^2}\frac{t_0-t}{4m^2}\tilde{H}^q_T. \quad \text{(A8)}$$

Here $\varepsilon=\pm 1$ is a sign factor whose value is not important for the derivation of the bounds on GPDs and t_0 is defined in Eq. (6).

Using the above expressions for $A_{\lambda'\mu',\lambda\mu}$ one finds the following nonzero matrix elements $\tilde{A}_{ab}$ (14):

$$\tilde{A}_{11}=\sqrt{1-\xi^2}\left[H^q+\tilde{H}^q+2H^q_T+\frac{-\xi^2(E^q+\tilde{E}^q+2E^q_T)+2\xi\tilde{E}^q_T}{1-\xi^2}+\frac{t_0-t}{2m^2}\tilde{H}^q_T\right], \quad \text{(A9)}$$

$$\tilde{A}_{22}=\sqrt{1-\xi^2}\left[H^q-\tilde{H}^q-\frac{\xi^2}{1-\xi^2}(E^q-\tilde{E}^q)+\frac{t_0-t}{2m^2}\tilde{H}^q_T\right], \quad \text{(A10)}$$

$$\tilde{A}_{12}=\varepsilon\frac{\sqrt{t_0-t}}{m}\left[\tilde{H}^q_T+(1-\xi)\frac{E^q_T+\tilde{E}^q_T}{2}+\frac{E^q-\xi\tilde{E}^q}{2}\right], \quad \text{(A11)}$$

$$\tilde{A}_{21}=\varepsilon\frac{\sqrt{t_0-t}}{m}\left[-\tilde{H}^q_T-(1+\xi)\,\frac{E^q_T-\tilde{E}^q_T}{2}-\frac{E^q+\xi\tilde{E}^q}{2}\right], \quad \text{(A12)}$$

$$\tilde{A}_{33}=\sqrt{1-\xi^2}\left[H^q+\tilde{H}^q-2H^q_T+\frac{\xi^2(-E^q-\tilde{E}^q+2E^q_T)-2\xi\tilde{E}^q_T}{1-\xi^2}-\frac{t_0-t}{2m^2}\tilde{H}^q_T\right], \quad \text{(A13)}$$

$$\tilde{A}_{44}=\sqrt{1-\xi^2}\left[H^q-\tilde{H}^q-\frac{\xi^2}{1-\xi^2}(E^q-\tilde{E}^q)-\frac{t_0-t}{2m^2}\tilde{H}^q_T\right], \quad \text{(A14)}$$

$$\tilde{A}_{34}=\varepsilon\frac{\sqrt{t_0-t}}{m}\left[\tilde{H}_T^q+(1-\xi)\frac{E_T^q+\tilde{E}_T^q}{2}-\frac{E^q-\xi\tilde{E}^q}{2}\right], \tag{A15}$$

$$\tilde{A}_{43}=-\varepsilon\frac{\sqrt{t_0-t}}{m}\left[\tilde{H}_T^q+(1+\xi)\frac{E_T^q-\tilde{E}_T^q}{2}-\frac{E^q+\xi\tilde{E}^q}{2}\right]. \tag{A16}$$

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